Differential equation 7.9.12

Ex 7

b). $y''-2y'+2y = t\cos t\, \mathrm{ch}(t)$

pf: $(y''-2y'+2y) = \left(\frac{d}{dt}-(1+i)\right)\left(\frac{d}{dt}-(1-i)\right)y = t\cos t\, \mathrm{ch}(t)$

g). $t(t+1)y''+(t+2)y'-y=0$

$\Rightarrow ((t+1)\frac{d}{dt}-1)(t\frac{d}{dt}+1)y=0$

Ex 9. $f''(t)=f(-t)$, $f\in C^2$

$$\begin{cases}(f(t)+f(-t))'' = f(t)+f(-t)\\ (f(t)-f(-t))'' = -(f(t)-f(-t))\end{cases}$$

$\Rightarrow f(t)+f(-t) = Ae^{t}+Be^{-t}$ $\xRightarrow{f(t)+f(-t) \text{ is even}}$ $f(t)+f(-t) = A(e^{t}+e^{-t})$

$f(t)-f(-t) = C\cos t + D\sin t \Rightarrow f(t)-f(-t) = D\sin t$

$\Rightarrow f(t) = C_1\cosh t + C_2\sin t$

Ex 12. $\lim_{t\to+\infty} y''+y'+y=0$

Let $g = y'+\frac{1+\sqrt{3}i}{2}y$, then

$$\lim_{t\to+\infty} g'-\frac{-1+\sqrt{3}i}{2}g = 0 .$$

Let $\alpha(t) = g'-\frac{-1+\sqrt{3}i}{2}g$, then

$$\frac{d}{dt}\left(e^{\frac{1-\sqrt{3}i}{2}t}g(t)\right) = e^{\frac{1-\sqrt{3}i}{2}t}\alpha(t)$$

$$\Rightarrow g(t) = e^{-\frac{1-\sqrt{3}i}{2}t}\left(C+\int_0^t e^{\frac{1-\sqrt{3}i}{2}s}\alpha(s)\,ds\right)$$

$$= e^{-\frac{1-\sqrt{3}i}{2}t}\left(C+\int_0^{t_0} e^{\frac{1-\sqrt{3}i}{2}s}\alpha(s)\,ds\right) + e^{-\frac{1-\sqrt{3}i}{2}t}\int_{t_0}^{t} e^{\frac{1-\sqrt{3}i}{2}s}\alpha(s)\,ds$$

$\forall \varepsilon>0$, $\exists\, t_0>0$ s.t. $\forall s\geq t_0$, we have $|\alpha(s)|<\varepsilon$

$$\Rightarrow \left|e^{-\frac{1-\sqrt{3}i}{2}t}\int_{t_0}^{t} e^{\frac{1-\sqrt{3}i}{2}s}\alpha(s)\,ds\right|$$

$$\leq \varepsilon\ e^{-\frac{t}{2}} \int_{t_0}^{t} e^{\frac{s}{2}} ds$$

$$= \varepsilon\ e^{-\frac{t}{2}} \cdot 2 (e^{\frac{t}{2}} - e^{\frac{t_0}{2}}) < 2\varepsilon$$

$$\Rightarrow \lim_{t \to +\infty} |g(t)| < 2\varepsilon$$

$$\Rightarrow \lim_{t \to +\infty} g(t) = 0$$

In a similar way, $\lim_{t \to +\infty} y(t) = 0$. $\square$

Differential Equation : 7, 9, 12

Ex 7

b). $y'' - 2y' + 2y = t\cdot \cos t \cdot \operatorname{ch} t$

① $\left(\frac{d}{dt} - (1+i)\right)\left(\frac{d}{dt} - (1-i)\right) y = t \cdot \frac{e^{it} + e^{-it}}{2} \cdot \frac{e^{t} + e^{-t}}{2}$

$= \frac{t}{4}\left(e^{(1+i)t} + e^{(1-i)t} + e^{(-1+i)t} + e^{(-1-i)t}\right)$

② Let $x(t) = y'(t)$, then

$$x'(t) - 2x(t) + 2y(t) = t\cdot \cos t \cdot \operatorname{ch} t$$

$$y'(t) - x(t) = 0$$

$$\Leftrightarrow \quad \frac{d}{dt}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} + \begin{pmatrix} -2 & 2 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} t\cdot\cos t\cdot \operatorname{ch} t \\ 0 \end{pmatrix}$$

$$\Leftrightarrow \quad \frac{d}{dt}\left(e^{\begin{pmatrix} -2 & 2 \\ -1 & 0 \end{pmatrix}t}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}\right) = e^{\begin{pmatrix} -2 & 2 \\ -1 & 0 \end{pmatrix}t}\begin{pmatrix} t\cdot\cos t\cdot \operatorname{ch} t \\ 0 \end{pmatrix}$$

$$e^{At} := \sum_{n=0}^{+\infty} \frac{(At)^n}{n!}$$

$$\Leftrightarrow \quad \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{-\begin{pmatrix} -2 & 2 \\ -1 & 0 \end{pmatrix}t}\left(C + \int_0^t e^{\begin{pmatrix} -2 & 2 \\ -1 & 0 \end{pmatrix}s}\begin{pmatrix} s\cdot\cos s\cdot \operatorname{ch} s \\ 0 \end{pmatrix}\right)$$

g). $t\cdot(t+1)y'' + (t+2)y' - y = 0$

① $\left((t+1)\frac{d}{dt} - 1\right)\left(t\frac{d}{dt} + 1\right) y = 0$

② Suppose that $y = \sum_{n=-\infty}^{+\infty} a_n t^n$, then

$$t\cdot(t+1)\cdot\sum_{n=-\infty}^{+\infty} a_n\cdot n\cdot(n-1)\,t^{n-2} + (t+2)\sum_{n=-\infty}^{+\infty} a_n\cdot n\cdot t^{n-1} - \sum_{n=-\infty}^{+\infty} a_n t^n = 0$$

$$\sum_{n=-\infty}^{+\infty} a_n\cdot n(n-1)\,t^n + \sum_{n=-\infty}^{+\infty} a_{n+1}(n+1)\cdot n\cdot t^n + \sum_{n=-\infty}^{+\infty} a_n\cdot n\cdot t^n + 2\sum_{n=-\infty}^{+\infty} a_{n+1}(n+1)\cdot t^n - \sum_{n=-\infty}^{+\infty} a_n t^n = 0$$

$$\Leftrightarrow \quad \left(n(n-1) + n - 1\right)a_n + \left((n+1)n + 2(n+1)\right)a_{n+1} = 0, \quad \forall n \in \mathbb{Z}$$

$\Leftrightarrow \quad (n-1)(n+1)a_n + (n+1)(n+2)a_{n+1} = 0, \ \forall n \in \mathbb{Z} \qquad (*)$

$\Leftrightarrow \quad n=1, \ a_2 = 0$

$n=0, \ a_0 = 2a_1$

$n=-1$, $(*)$ always hold.

$n=-2, \ a_{-2} = 0$

$n \geq 2, \ a_{n+1} = \frac{1-n}{n+2} a_n = 0$

$n \leq -3, \ a_n = \frac{n+2}{1-n} a_{n+1} = 0$

$\Leftrightarrow \quad y = a_{-1} \cdot \frac{1}{t} + 2a_1 + a_1 t = a_{-1} \cdot \frac{1}{t} + a_1(2+t)$

$$\left[\ f(x) = \begin{cases} e^{-\frac{1}{x^2}} & x \neq 0 \\ 0 & x = 0 \end{cases} \qquad f^{(n)}(0) = 0 \ \text{ but } \ f(x) \neq 0 \ \right]$$

Ex 9. $f''(t) = f(-t), \ f \in C^2$

$$\begin{cases} (f(t)+f(-t))'' = f(t)+f(-t) \\ (f(t)-f(-t))'' = -(f(t)-f(-t)) \end{cases} \Leftrightarrow \begin{array}{l} f(t)+f(-t) = Ae^{t}+Be^{-t} \ \& \ B = A \\ f(t)-f(-t) = C\cos t + D\sin t \ \& \ C = 0 \end{array}$$

$\Leftrightarrow \quad f(t) = C_1 \cosh t + C_2 \sin t$

Ex 12. $\lim_{t\to+\infty} y''+y'+y = 0$

$\left(\frac{d}{dt} - \frac{-1+\sqrt{3}i}{2}\right)\left(\frac{d}{dt} - \frac{-1-\sqrt{3}i}{2}\right) y = \alpha(t), \quad \lim_{t\to+\infty} \alpha(t) = 0$

Let $x(t) = y'(t) - \frac{-1-\sqrt{3}i}{2} y(t)$, then

$$x'(t) - \frac{-1+\sqrt{3}i}{2} x(t) = \alpha(t)$$

$\Rightarrow \quad \frac{d}{dt}\left(e^{-\frac{-1+\sqrt{3}i}{2}t} x(t)\right) = e^{-\frac{-1+\sqrt{3}i}{2}t} \alpha(t)$

$\Rightarrow \quad x(t) = e^{\frac{-1+\sqrt{3}i}{2}t}\left(C + e^{\frac{-1+\sqrt{3}i}{2}t} \int_0^t e^{-\frac{-1+\sqrt{3}i}{2}s} \alpha(s)\, ds\right).$

Claim: $\lim_{t\to+\infty} x(t) = 0$

$\forall \varepsilon > 0$, $\exists\, t_0$ s.t. $|\alpha(s)| < \varepsilon$, $\forall s > t_0$, then

$$|\gamma(t)| \le C \cdot e^{-\frac{t}{2}} + e^{-\frac{t}{2}} \cdot \int_0^t e^{\frac{s}{2}} |\alpha(s)|\, ds$$

$$\le \left(C + \int_0^{t_0} e^{\frac{s}{2}} |\alpha(s)|\, ds\right) e^{-\frac{t}{2}} + \varepsilon \cdot e^{-\frac{t}{2}} \int_0^t e^{\frac{s}{2}}\, ds$$

$$= \left(C + \int_0^{t_0} e^{\frac{s}{2}} |\alpha(s)|\, ds\right) e^{-\frac{t}{2}} + \frac{\varepsilon}{2} \underbrace{\left(1 - e^{-\frac{t}{2}}\right)}_{<1}$$

$\exists\, t_1$ s.t. $\forall t > t_1$

$$\le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

□